# Matrix Description of M-theory on $T^4$ and $T^5$

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We study the Matrix theory description of M-theory compactified on $T^4$ and $T^5$. M-theory on $T^4$ is described by the six dimensional (2,0) fixed point field theory compactified on a five torus, $\widetilde{T}^5$. For M-theory on $T^5$ we suggest the existence of a new theory which is compactified on a $\widetilde{T}^5$. The IR description of this theory is given by the (2,0) theory with a compactified moduli space. This new theory appears to be a new kind of a non-critical string theory. Clearly, these two descriptions differ from the "standard" Super-Yang-Mills on the dual torus prescription.

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# 1. Introduction

In the last few months a significant amount of evidence has accumulated in support of the conjecture of Banks, Fischler, Shenker and Susskind on the non-perturbative formulation of M-theory [1]. The structure of 11-dimensional supergravity and the membrane of M-theory were found in [1]. The longitudinal 5-brane was found in [2-4]. Various D-branes were constructed in [4]. Elementary strings and part of their interactions were constructed in [5-8]. The interactions of solitonic states also seem to fit the expected pattern from M-theory [9].

In [1-4,6,10-14] compactifications on tori were considered. It was suggested that M theory on $T^d$ is defined by $d + 1$ dimensional super-Yang-Mills (SYM) on $\hat{T}^d$ where $\hat{T}^d$ is a torus dual to the space-time torus. However, for $d > 3$ these field theories are not renormalizable, and therefore cannot give a complete description of the theory. Furthermore, one cannot define them as the long distance limit of another theory in $d + 1$ dimensions with the same amount of supersymmetry. For such a description to exist, there must be a fixed point of the renormalization group in $d + 1$ dimensions with that supersymmetry. However, such fixed points do not exist [15].

It is possible that these theories can be defined by a fixed point with fewer supersymmetries. Another approach, which we will pursue here, is to *define* the 4+1 dimensional theory not by a 4+1 dimensional fixed point but by a 5+1 dimensional fixed point with (2,0) supersymmetry [16,17]. This leads to a geometric understanding of the U-duality group [17,14]. It is important to stress that the 5+1 dimensional description does not follow from the 4+1 dimensional gauge theory. It is a definition of the latter. We thus propose to define M theory on $T^4$ as the (2,0) fixed point theory in six dimensions compactified on $\widetilde{T}^5$. The precise five torus $\widetilde{T}^5$ and the details of this construction will be presented in section 2.

Our work has implications to the nature of the underlying space-time. In the SYM prescription space-time appears as the moduli space of vacua of the theory reduced to quantum mechanics. Strictly speaking, in quantum mechanics the notion of moduli space of vacua is ill defined. If the theory has a classical limit, this moduli space is the target space of the theory in this limit. In the example of M theory on $T^4$, the (2,0) theory is inherently quantum mechanical – the two form whose field strength is self-dual forces $\hbar$ to be fixed at order one. Therefore, this theory does not have a classical limit and hence we cannot define the moduli space of its vacua, when it is compactified on $\widetilde{T}^5$. However, if one of the circles of this $\widetilde{T}^5$ is much smaller than the others, we have an approximate 4+1 dimensional SYM with a fixed $\hbar$. This theory has a well defined moduli space of vacua.

We see that the notion of space-time becomes well defined only at various boundaries of the $\widetilde{T}^5$ parameter space. We will explain this in more detail at the end of section 2.

For M-theory on $T^5$ we do not have a full description of the theory. We suggest that it is given by a new theory whose effective IR description is given in terms of the (2,0) theory with a modified moduli space, very much like the description of [18], although for somewhat different reasons. In the process we identify all the 16 states that correspond to M-theory wrapped membranes, wrapped 5-brane and Kaluza-Klein modes. In a way which is somewhat reminiscent of the work of [18], we also suggest that in this case the full theory can perhaps be described as a new kind of non-critical string theory on $\widetilde{T}^5$. This explains the origin of the $SO(5,5,\mathbb{Z})$ U-duality group as T-duality of this theory. This will be discussed in section 3.

These two proposals present a departure from the SYM prescription of the Matrix model. For the theory on $T^4$ we use a field theory which is not fully understood. For the theory on $T^5$ our prescription is clearly incomplete. For higher dimensional tori we expect new elements to come in. In particular, the exceptional U-duality groups of these theories makes them especially interesting.

## 2. M Theory on $T^4$

As a first attempt to define a Matrix Theory description of M-theory compactified on $T^4$ one studies 4+1 dimensional SYM theory compactified on the dual torus $\hat{T}^4$. Its field content is one vector multiplet in the adjoint of $U(N)$. For simplicity we will take the torus to be rectangular. M-theory has five dimensionful parameters which are the Planck length, $l_p$, and the periodicities of the torus, $L_i$. The parameters of the gauge theory are the dual torus lengths $\widetilde{L}_i$ and the gauge coupling $g^2$. They are given by [17,14]:

$$
\begin{aligned}
\widetilde{L}_i &= \frac{(2\pi)^2 l_p{}^3}{L_i R} \\
g^2 &= \frac{(2\pi)^6 l_p{}^6}{L_1 L_2 L_3 L_4 R}
\end{aligned}
\tag{2.1}
$$

where $R$ is the radius of the compactified longitudinal direction.

This gauge theory is not renormalizable and therefore not well defined. However, it can be used as a low energy effective theory which is valid at energies below $\frac{1}{g^2}$. In this regime several tests of the description of this theory as a formulation of M-theory were given. These include the identification of the various fluxes in the theory and their relations to the various BPS charges. Also, the massive "W-bosons" of this theory have the right mass (as a function of the moduli and the radii) to give the proper graviton scattering.

To define our 4+1 dimensional nonrenormalizable theory we need to give more information about its short distance degrees of freedom. On general grounds, it is impossible to find these UV degrees of freedom using only the IR information. In our case, we are guided by the $SL(5, \mathbb{Z})$ U-duality group to suggest that the desired definition of this theory is in terms of the $(2, 0)$ supersymmetric fixed point in six dimensions.

The $(2, 0)$ theory is an interacting quantum field theory at a fixed point of the renormalization group (for a review, see [15]). It was first discussed in the context of type IIB compactification on a singular K3 [19] and later in the context of $N$ nearby 5-branes in M-theory [20,21]. Here we use the same field theory as a definition of M-theory on $T^4$. We should stress, to avoid confusion, that we are not defining M theory as the theory of 5-branes. We simply use the theory on the 5-brane as an analogue model for the $(2,0)$ field theory. It is this field theory which we use. This distinction will become more clear below.

We thus propose that M-theory on $T^4$ with radii $L_{1,2,3,4}$ is described by the large $N$ limit of the $(2,0)$ theory compactified[1] on a five torus $\widetilde{T}^5$. Its five sizes are related to $L_{1,2,3,4}$ in the following way. In (2.1) the sizes of the four torus and the coupling of the 4+1 SYM were given. This SYM has an additional conserved $U(1)$ current given by $j = {}^*(F \wedge F)$. This symmetry is to be identified with the Kaluza-Klein $U(1)$ symmetry of rotating around the small circle. The circumference of this circle is $\frac{g^2}{2\pi}$. In the 4+1 SYM an $n$ instanton configuration, which has energy $\frac{4\pi^2 n}{g^2}$, corresponds to a state with total momentum $n$ around the small circle and hence we identify its periodicity with $\frac{g^2}{2\pi}$. The 4+1 SYM description only identifies the charge of this $U(1)$ symmetry. It cannot answer questions regarding the detailed nature of these instantons such as their bound states, etc.. Such questions can be answered only in a complete description of the short distance theory.

To summarize, we propose that (2,0) theory is compactified on $\widetilde{T}^5$ whose sizes are given by

$$\widetilde{L}_i = \frac{(2\pi)^2 l_p{}^3}{L_i R}$$
$$\widetilde{L}_5 = \frac{(2\pi)^5 l_p{}^6}{L_1 L_2 L_3 L_4 R}. \tag{2.2}$$

As a first test of this proposal we point out that at energies much smaller than $\frac{1}{g^2} = \frac{1}{2\pi \widetilde{L}_5}$ and for $\widetilde{L}_5 \ll \widetilde{L}_{1,2,3,4}$ our theory becomes the 4+1 dimensional SYM. This follows from the fact that the dimensional reduction of the two form $B$ whose field strength $H = dB$ is selfdual is a vector in five dimensions. No scalar emerges from this dimensional reduction, and therefore the moduli space of vacua of the compactified (2,0) theory is

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[1] E. Witten [22] pointed out that this theory needs more data for its definition. We are not sure how this affects our proposal.

the same as that of the gauge theory. We will discuss the details of the moduli space of vacua below. Furthermore, the excitations of this six dimensional theory include strings. As they wind around the circle which brings them to five dimensions, they yield the W bosons of the 4+1 dimensional SYM. Their mass is given correctly by the expression in the (2,0) theory, and their contribution to the graviton scattering amplitude is as in the SYM. However, the 4+1 dimensional SYM is not exact. Kaluza-Klein excitations of the 5+1 dimensional theory are also important and lead to crucial differences between our theory and the 4+1 SYM theory. For example, the Kaluza-Klein excitations contribute to graviton scattering terms which are suppressed by the volume of the space-time $T^4$. Note that strings which do not wind around the circle are not new degrees of freedom in the 4+1 SYM. They are given by classical solutions (like four dimensional monopoles) in this theory.

A less trivial test of this proposal [17] is that this definition of M-theory on $T^4$ makes the U-duality manifest. The U-duality group in this case is $SL(5, \mathbb{Z})$. It is simply the geometric duality group of $\widetilde{T}^5$. This symmetry involves mixing the five radii $\widetilde{L}$ in a way which is complicated as action on the individual $L_i$. Since this U-duality group is manifest, so are its subgroups which appear in compactifications on lower dimensional tori.

We can also compare the parameters of M-theory with the parameters of this model. In M-theory there is a single dimensionful parameter, $l_p$, and 14 dimensionless parameters. These include 10 metric parameters and 4 $C$ field parameters (these include various parameters which we previously set to zero for simplicity). In the Matrix theory there seem to be 15 parameters in the metric of the 5-torus. However, since the (2,0) theory is scale invariant, only 14 of them are meaningful – the 15th sets the scale. We identify the 14 dimensionless parameters with the 14 dimensionless parameters of the M-theory compactification.

The various BPS charges which are scalars in the noncompact dimensions can be identified as fluxes in the Matrix theory. The fluxes in the 4+1 dimensional SYM are 4 electric fluxes and 6 magnetic ones, all living in the overall $U(1)$ factor of the $U(N)$ gauge theory. These fluxes correspond to momentum modes of 0-branes on the 4-torus, and membranes wrapped around 2-cycles of the 4-torus. Our 5+1 dimensional theory allows us to identify them as the 10 fluxes $\int dB$ on 3 cycles of our base $\widetilde{T}^5$. This expression makes the $SL(5, \mathbb{Z})$ transformation properties of the fluxes manifest.

In the compactified SYM description space-time emerges as the moduli space of vacua of the quantum mechanical system. It is easy to see that no such description is possible in the (2,0) theory. In fact, the theory of a single two form $B$, whose field strength $H = dB$ is self-dual on $\widetilde{T}^5$, has no moduli space of vacua for the $B$ field. As in the SYM case, we could

attempt to construct the light degrees of freedom on the moduli space by considering the constant modes of $B$. The self-duality condition forces these modes to be time independent and hence the theory has no light modes and no moduli space of vacua.

In fact, in quantum mechanics (unlike field theory with more than 2 space-time dimensions) there is never a moduli space of vacua. Only if the theory has a parameter, which can be interpreted as $\hbar$, can we expect an approximate notion of moduli space of vacua. Then, for $\hbar = 0$ the classical theory can have many static solutions which we can identify as its moduli space of vacua, $\mathcal{M}$. When $\hbar$ is small we can study the full quantum theory by restricting the degrees of freedom to $\mathcal{M}$ and quantizing only them (Born-Oppenheimer approximation).

Returning to our (2,0) theory on $\widetilde{T}^5$, we realize that this theory does not have a free parameter like $\hbar$. The six dimensional theory, because of its self-duality, has fixed $\hbar = 1$. Hence, as we saw above, it cannot have a semiclassical limit with a moduli space of vacua. Instead, we can find a moduli space of vacua by creating an effective $\hbar$. One way to do that is to consider the limit of this theory with $\widetilde{L}_5 \ll \widetilde{L}_{1,2,3,4}$. Then, by going through the 4+1 dimensional SYM we find a quantum mechanical system whose moduli space is $T^4$ with sizes $L_{1,2,3,4} \sim \frac{1}{\widetilde{L}_{1,2,3,4}}$. This interpretation of space-time is not natural, if we pursue it to the region where all the $\widetilde{L}$'s are of the same order of magnitude. As any one of the $\widetilde{L}$'s becomes much smaller than the others there is another natural interpretation of space time. This is the essence of U-duality in our construction. The natural interpretation of the theory in terms of space time changes as the five radii $\widetilde{L}$ change.


## 3. M-Theory on $T^5$

In this section we will attempt to define a Matrix model prescription for M-theory on $T^5$ with circumferences $L_{1,2,3,4,5}$. Our proposal is more speculative than the one in the previous section. Already in the compactification on $T^4$ we had to depart from the "SYM on the dual torus" prescription because this theory is not renormalizable. Clearly, for $T^5$ the problem is even worse. First, there is no fixed point which flows to SYM in six dimensions [15]. Furthermore, the description of M-theory on $T^5$ should include, as one of the radii goes to infinity, the discussion in the previous section. Therefore, the theory which is relevant for M-theory on $T^5$ should be such that it can flow to the (2,0) theory on $\widetilde{T}^5$ with circumferences

$$\widetilde{L}_i = \frac{(2\pi)^2 l_p^{\ 3}}{L_i R}$$

$$\widetilde{L}_5 = \frac{(2\pi)^5 l_p^{\ 6}}{L_1 L_2 L_3 L_4 R}.$$

(3.1)

This fact shows that the desired theory cannot be a field theory in more than six dimensions – in higher than six dimensions there is no supersymmetry algebra which includes the six dimensional (2,0) algebra as a subalgebra.

We do not have a complete suggestion for the full theory that describes M-theory on $T^5$. Instead, we will present an auxiliary model that shares the same IR behavior as this theory. It is the theory of $N$ M-theoretic 5-branes wrapping 5 cycles of a 6-torus. The circumferences of the wrapped circles are $\widetilde{L}_{1,2,3,4,5}$ of (3.1) and the sixth circumference is $L_5$. This description is particularly useful when the sixth circle is very large. It should be stressed that this is merely an analogue model. The IR behavior of our six dimensional theory is the same as the IR behavior of this 5-brane theory. We should not attempt to identify the base space of our six dimensional theory as the world volume of a 5-brane in M theory.

One way to motivate this claim is to start with M-theory on $T^5$ with one of its radii very large, $L_5 \gg L_{1,2,3,4}$. Then, the description of M-theory on $T^4$ should be approximately correct. As in the previous section, this is given by the (2,0) theory on $\widetilde{T}^5$. Now, we try to add the effects of the other circle. We look for a theory which in the limit $L_5 \to \infty$ flows to the (2,0) theory. Furthermore, its moduli space of vacua should have a circle of radius $L_5$. Our proposal in the previous paragraph satisfies these requirements.

The analogue model has finite tension strings which are M-theory membranes stretching from one 5-brane to another around the compact circle. We expect the full theory to have these strings also. To see this, consider the following process. Start with $N$ nearby 5-branes. They are described by the (2,0) fixed point. When they are not on top of each other the spectrum includes strings. These originate from membranes which are stretched between them. The tension of these strings vanishes as the 5-branes approach each other. There are also strings whose tension is proportional to the circumference of the compact scalar, $L_5$. They originate from membranes which wind around the circle. Now, move on the moduli space of vacua by moving one of the 5-branes around the circle and bringing it close to the other $N - 1$ 5-branes from the other side. The strings which were nearly tensionless at the beginning of the process now have tension of order $L_5$. However, new strings become nearly tensionless now. These originate from membranes which were previous wound around the circle. Since they are light, they cannot be ignored. They must also be included in our theory, if it has the same IR behavior as the analogue model.

Note, that unlike the fixed point with (2,0) supersymmetry which is scale invariant, this theory has a scale, $L_5$. It is the tension of strings which are not light near the singularity. Below this scale we have the (2,0) field theoretic degrees of freedom and one scalar is compact. Above it new degrees of freedom (those which make the strings) become important.

The effective IR description that we have obtained is similar to that of [18] and we can identify the fluxes in the same manner and show that their energies matches the 16 M-theoretic states that form the spinor of the $SO(5,5,\mathbb{Z})$ duality group. From the M-theory point of view, the 16 dimensional spinor representation contains 5 momentum modes in the compact directions, 10 membranes wrapped around 2-cycles of $T^5$ and the completely wrapped transverse 5-brane. Fluxes which are independent of the compact scalar, $\phi$, exist already when compactifying on a 4-torus and were discussed in the previous section. The rest of the states utilize the compact scalar. As in [18] the corresponding fluxes are $\partial_0 \int_{\widetilde{T}^5} \phi$ and $\oint d\phi$ over the five 1-cycles of the torus. The $\phi$ dependent terms of the Hamiltonian are

$$H = \frac{R}{2\widetilde{V}N} \int d^5\sigma {\Pi_\phi}^2 + \frac{1}{2R\widetilde{V}N} \int d^5\sigma \partial_i\phi\partial^i\phi, \qquad (3.2)$$

where $\widetilde{V}$ is the volume of $\widetilde{T}^5$ and $\Pi_\phi$ is the "overall U(1)" momentum conjugate to $\phi$. Its zero mode is quantized, $\Pi_5^0 = \frac{2\pi}{L_5}$. The quantum of energy is then:

$$E = \frac{R}{2N}(\Pi_\phi^0)^2 = \frac{(2\pi)^2 R}{2N{L_5}^2}. \qquad (3.3)$$

The corresponding mass in spacetime is $M = \frac{2\pi}{L_5}$. The other fluxes, associated with $\oint d\phi$ lead to new conserved charges. Denoting the base coordinates by $\sigma_i$, charged states are of the form:

$$\phi(\sigma_i) = \frac{n_i L_5}{\widetilde{L}_i}\sigma_i. \qquad (3.4)$$

The energy of these 5 "winding modes" can be computed from (3.2). This leads to the masses

$$M_i = \frac{L_5}{\widetilde{L}_i R} = \frac{L_5 L_i}{(2\pi)^2 {l_p}^3}, \quad i = 1,\ldots,4 \qquad (3.5)$$

and

$$M = \frac{L_5}{\widetilde{L}_5 R} = \frac{L_1 L_2 L_3 L_4 L_5}{(2\pi)^5 {l_p}^6}. \qquad (3.6)$$

The first four states can be identified with membranes wrapped around 2 dimensions of the spacetime $T^5$ and the fifth corresponds to the completely wrapped transverse 5-brane.

Since, as we discussed above, our theory has finite tension strings, it is tempting to interpret the U-duality group $SO(5,5,\mathbb{Z})$ of the theory as the T-duality of these strings. This interpretation can be carried further. M-theory on $T^5$ has one dimensionful parameter, $l_p$, and 25 dimensionless parameters. 15 of them originate from the metric, and 10 from the $C$ field. A string theory on a 5-torus has the fundamental scale, the string scale (its $\alpha'$), 15 dimensionless metric parameters and 10 dimensionless parameters in the $B$

field that couples to that string. The $SO(5, 5, \mathbb{Z})$ T-duality group acts on these parameters. Perhaps the full theory which describes M-theory on $T^5$ is the large $N$ limit of some non-critical string theory.

# References

[1] T. Banks, W. Fischler, S. Shenker and L. Susskind, hep-th/9610043, *Phys. Rev.* **D55,** 112 (1997).

[2] M. Berkooz and M.R. Douglas, hep-th/9610236, *Phys. Lett.* **B395,** 196 (1997).

[3] O.J. Ganor, S. Ramgoolam and W. Taylor IV, hep-th/9611202.

[4] T. Banks, N. Seiberg and S.H. Shenker, hep-th/9612157, *Nucl. Phys.* **B490,** 91 (1997).

[5] L. Motl, hep-th/9701025.

[6] T. Banks and N. Seiberg, hep-th/9702187.

[7] P.-M. Ho and Y.-S. Wu, hep-th/9703016.

[8] R. Dijkgraaf, E. Verlinde and H. Verlinde, hep-th/9703030.

[9] O. Aharony and M. Berkooz, hep-th/9611215; G. Lifschytz and Samir D. Mathur, hep-th/9612087; G. Lifschytz, hep-th/9612223; D. Berenstein and R. Corrado, hep-th/9702108; G. Lifschytz, hep-th/9703201; I. Cheplev and A.A. Tseytlin, hep-th/9704127.

[10] W. Taylor IV, hep-th/9611042, *Phys. Lett.* **B394,** 283 (1997).

[11] L. Susskind, hep-th/9611164.

[12] S. Sethi and L. Susskind, hep-th/9702101.

[13] D. Berenstein, E. Corrado and J. Distler, hep-th/9704087.

[14] W. Fischler, E. Halyo, A. Rajaraman and L. Susskind, hep-th/9703102.

[15] N. Seiberg, "Notes on theories with 16 supercharges," RU-97-7, to appear.

[16] N. Seiberg, talk given at the Jerusalem Winter School on Strings and Duality, Jan. 1997.

[17] M. Rozali, hep-th/9702136.

[18] R. Dijkgraaf, E. Verlinde and H. Verlinde, hep-th/9603126, *Nucl. Phys.* **B486,** 77 (1997); hep-th/9604055, *Nucl. Phys.* **B486,** 89 (1997).

[19] E. Witten, hep-th/9507121, Contributed to STRINGS 95: Future Perspectives in String Theory, Los Angeles, CA, 13-18 Mar 1995.

[20] A. Strominger, hep-th/9512059, *Phys. Lett.* **B383,** 44 (1996).

[21] E. Witten, hep-th/9512219, *Nucl. Phys.* **B463,** 383 (1996).

[22] E. Witten, hep-th/9610234.